EXAMPLE 6 Combining Transformations in Order

(a) The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.

- a horizontal shift 2 units to the right
- a vertical stretch by a factor of 3
- a vertical translation 5 units up

(b) Apply the transformations in (a) in the opposite order and find the equation of the graph that results.

SOLUTION

(a) Applying the transformations in order, we have

$$x^2 \Rightarrow (x-2)^2 \Rightarrow 3(x-2)^2 \Rightarrow 3(x-2)^2 + 5$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 17$.

(b) Applying the transformations in the opposite order, we have

$$x^2 \Rightarrow x^2 + 5 \Rightarrow 3(x^2+5) \Rightarrow 3((x-2)^2 + 5)$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 27$.

The second graph is ten units higher than the first graph because the vertical stretch lengthens the vertical translation when the translation occurs first. Order often matters when stretches, shrinks, or reflections are involved.

Now try Exercise 47.

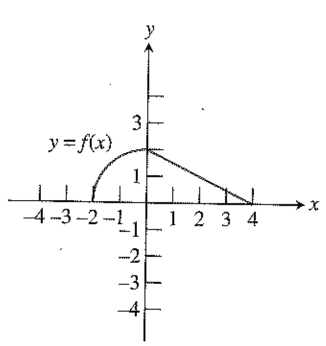


FIGURE 1.78 The graph of the function $y = f(x)$ in Example 7.

EXAMPLE 7 Transforming a Graph Geometrically

The graph of $y = f(x)$ is shown in Figure 1.78. Determine the graph of the composite function $y = 2f(x+1) - 3$ by showing the effect of a sequence of transformations on the graph of $y = f(x)$.

SOLUTION

The graph of $y = 2f(x+1) - 3$ can be obtained from the graph of $y = f(x)$ by the following sequence of transformations:

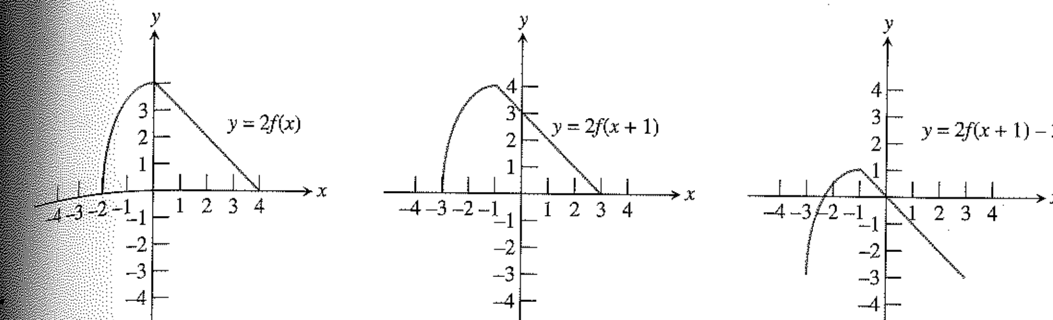
(a) a vertical stretch by a factor of 2 to get $y = 2f(x)$ (Figure 1.79a)

(b) a horizontal translation 1 unit to the left to get $y = 2f(x+1)$ (Figure 1.79b)

(c) a vertical translation 3 units down to get $y = 2f(x+1) - 3$ (Figure 1.79c)

(The order of the first two transformations can be reversed without changing the final graph.)

Now try Exercise 51.



Vertical stretch of factor 2

(a)

Horizontal translation left 1 unit

(b)

Vertical translation down 3 units

(c)

FIGURE 1.79 Transforming the graph of $y = f(x)$ in Figure 1.78 to get the graph of $y = 2f(x+1) - 3$. (Example 7)

QUICK REVIEW 1.6 *(For help, go to Section A.2.)*

In Exercises 1–6, write the expression as a binomial squared.

1. $x^2 + 2x + 1$
2. $x^2 - 6x + 9$
3. $x^2 + 12x + 36$
4. $4x^2 + 4x + 1$
5. $x^2 - 5x + \dfrac{25}{4}$
6. $4x^2 - 20x + 25$

In Exercises 7–10, perform the indicated operations and simplify.

7. $(x-2)^2 + 3(x-2) + 4$
8. $2(x+3)^2 - 5(x+3) - 2$
9. $(x-1)^3 + 3(x-1)^2 - 3(x-1)$
10. $2(x+1)^3 - 6(x+1)^2 + 6(x+1) - 2$

SECTION 1.6 EXERCISES

In Exercises 1–8, describe how the graph of $y = x^2$ can be transformed to the graph of the given equation.

1. $y = x^2 - 3$
2. $y = x^2 + 5.2$
3. $y = (x+4)^2$
4. $y = (x-3)^2$
5. $y = (100 - x)^2$
6. $y = x^2 - 100$
7. $y = (x-1)^2 + 3$
8. $y = (x+50)^2 - 279$

In Exercises 9–12, describe how the graph of $y = \sqrt{x}$ can be transformed to the graph of the given equation.

9. $y = -\sqrt{x}$
10. $y = \sqrt{x-5}$
11. $y = \sqrt{-x}$
12. $y = \sqrt{3-x}$

In Exercises 13–16, describe how the graph of $y = x^3$ can be transformed to the graph of the given equation.

13. $y = 2x^3$
14. $y = (2x)^3$
15. $y = (0.2x)^3$
16. $y = 0.3x^3$

In Exercises 17–20, describe how to transform the graph of f into the graph of g.

17. $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{x-4}$
18. $f(x) = (x-1)^2$ and $g(x) = -(x+3)^2$
19. $f(x) = (x-2)^3$ and $g(x) = -(x+2)^3$
20. $f(x) = |2x|$ and $g(x) = 4|x|$

In Exercises 21–24, sketch the graphs of f, g, and h by hand. Support your answers with a grapher.

21. $f(x) = (x+2)^2$
$g(x) = 3x^2 - 2$
$h(x) = -2(x-3)^2$

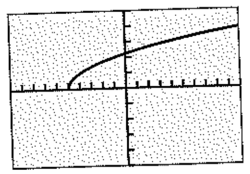
22. $f(x) = x^3 - 2$
$g(x) = (x+4)^3 - 1$
$h(x) = 2(x-1)^3$

23. $f(x) = \sqrt[3]{x+1}$
$g(x) = 2\sqrt[3]{x} - 2$
$h(x) = -\sqrt[3]{x-3}$

24. $f(x) = -2|x| - 3$
$g(x) = 3|x+5| + 4$
$h(x) = |3x|$

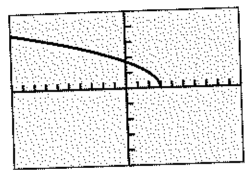
In Exercises 25–28, the graph is that of a function $y = f(x)$ that can be obtained by transforming the graph of $y = \sqrt{x}$. Write a formula for the function f.

25.



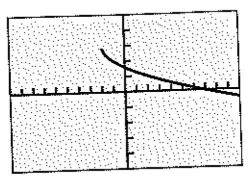
[–10, 10] by [–5, 5]

26.



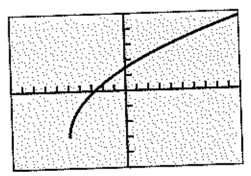
[–10, 10] by [–5, 5]

27.



[–10, 10] by [–5, 5]

28.



[–10, 10] by [–5, 5]
Vertical stretch = 2

In Exercises 29–32, find the equation of the reflection of f across **(a)** the x-axis and **(b)** the y-axis.

29. $f(x) = x^3 - 5x^2 - 3x + 2$ **30.** $f(x) = 2\sqrt{x+3} - 4$

31. $f(x) = \sqrt[3]{8x}$ **32.** $f(x) = 3|x + 5|$

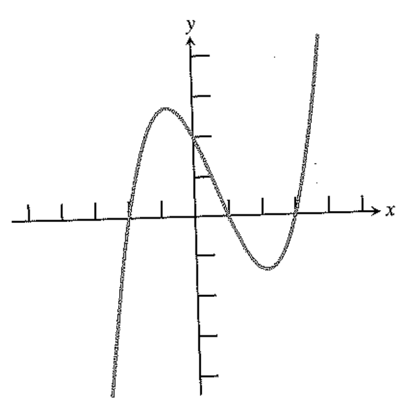
33. Reflecting Odd Functions Prove that the graph of an odd function is the same when reflected across the x-axis as it is when reflected across the y-axis.

34. Reflecting Odd Functions Prove that if an odd function is reflected about the y-axis and then reflected again about the x-axis, the result is the original function.

Exercises 35–38 refer to the graph of $y = f(x)$ shown below. In each case, sketch a graph of the new function.

35. $y = |f(x)|$ **36.** $y = f(|x|)$

37. $y = -f(|x|)$ **38.** $y = |f(|x|)|$



In Exercises 39–42, transform the given function by **(a)** a vertical stretch by a factor of 2, and **(b)** a horizontal shrink by a factor of 1/3.

39. $f(x) = x^3 - 4x$ **40.** $f(x) = |x + 2|$

41. $f(x) = x^2 + x - 2$ **42.** $f(x) = \dfrac{1}{x+2}$

In Exercises 43–46, describe a basic graph and a sequence of transformations that can be used to produce a graph of the given function.

43. $y = 2(x - 3)^2 - 4$ **44.** $y = -3\sqrt{x+1}$

45. $y = (3x)^2 - 4$ **46.** $y = -2|x + 4| + 1$

In Exercises 47–50, a graph G is obtained from a graph of y by the sequence of transformations indicated. Write an equation whose graph is G.

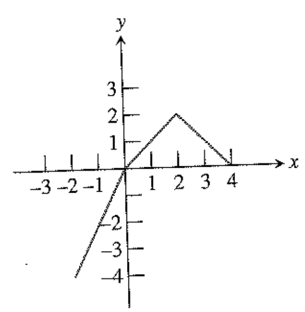
47. $y = x^2$: a vertical stretch by a factor of 3, then a shift right 4 units.

48. $y = x^2$: a shift right 4 units, then a vertical stretch by a factor of 3.

49. $y = |x|$: a shift left 2 units, then a vertical stretch by a factor of 2, and finally a shift down 4 units.

50. $y = |x|$: a shift left 2 units, then a horizontal shrink by a factor of 1/2, and finally a shift down 4 units.

Exercises 51–54 refer to the function f whose graph is shown below.



51. Sketch the graph of $y = 2 + 3f(x + 1)$.

52. Sketch the graph of $y = -f(x + 1) + 1$.

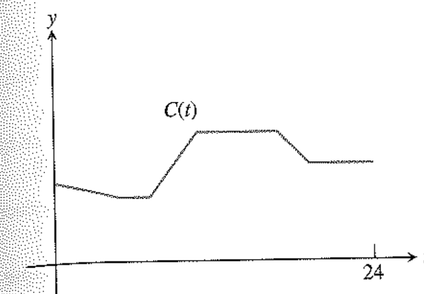
53. Sketch the graph of $y = f(2x)$.

54. Sketch the graph of $y = 2f(x - 1) + 2$.

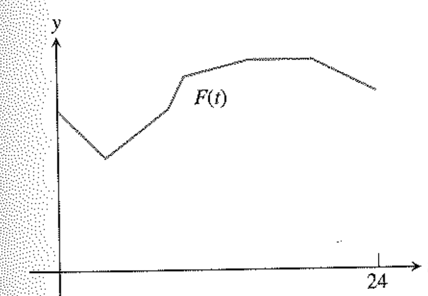
55. Writing to Learn Graph some examples to convince yourself that a reflection and a translation can have a different effect when combined in one order than when combined in the opposite order. Then explain in your own words why this can happen.

56. Writing to Learn Graph some examples to convince yourself that vertical stretches and shrinks do not affect a graph's x-intercepts. Then explain in your own words why this is so.

57. Celsius vs. Fahrenheit The graph shows the temperature in degrees Celsius in Windsor, Ontario, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Fahrenheit. [*Hint*: $F(t) = (9/5)C(t) + 32$.]



58. Fahrenheit vs. Celsius The graph shows the temperature in degrees Fahrenheit in Mt. Clemens, Michigan, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Celsius.
[*Hint*: $F(t) = (9/5)C(t) + 32$.]



Standardized Test Questions

59. True or False The function $y = f(x + 3)$ represents a translation to the right by 3 units of the graph of $y = f(x)$. Justify your answer.

60. True or False The function $y = f(x) - 4$ represents a translation down 4 units of the graph of $y = f(x)$. Justify your answer.

In Exercises 61–64, you may use a graphing calculator to answer the question.

61. Multiple Choice Given a function f, which of the following represents a vertical stretch by a factor of 3?

(A) $y = f(3x)$ **(B)** $y = f(x/3)$

(C) $y = 3f(x)$ **(D)** $y = f(x)/3$

(E) $y = f(x) + 3$

62. Multiple Choice Given a function f, which of the following represents a horizontal translation of 4 units to the right?

(A) $y = f(x) + 4$ **(B)** $y = f(x) - 4$

(C) $y = f(x + 4)$ **(D)** $y = f(x - 4)$

(E) $y = 4f(x)$

63. Multiple Choice Given a function f, which of the following represents a vertical translation of 2 units upward, followed by a reflection across the y-axis?

(A) $y = f(-x) + 2$ **(B)** $y = 2 - f(x)$

(C) $y = f(2 - x)$ **(D)** $y = -f(x - 2)$

(E) $y = f(x) - 2$

64. Multiple Choice Given a function f, which of the following represents reflection across the x-axis, followed by a horizontal shrink by a factor of 1/2?

(A) $y = -2f(x)$ **(B)** $y = -f(x)/2$

(C) $y = f(-2x)$ **(D)** $y = -f(x/2)$

(E) $y = -f(2x)$

Explorations

65. International Finance Table 1.11 shows the price of a share of stock in Dell Computer for the first eight months of 2004:

Table 1.11 Dell Computer

Month	Price ($)
1	33.44
2	32.65
3	33.62
4	34.78
5	35.24
6	35.82
7	35.47
8	34.84

Source: Yahoo! Finance

(a) Graph price (y) as a function of month (x) as a line graph, connecting the points to make a continuous graph.

(b) Explain what transformation you would apply to this graph to produce a graph showing the price of the stock in Japanese yen.

66. Group Activity Get with a friend and graph the function $y = x^2$ on both your graphers. Apply a horizontal or vertical stretch or shrink to the function on one of the graphers. Then change the *window* of that grapher to make the two graphs look the same. Can you formulate a general rule for how to find the window?